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Lattice Gas Generalization of the Hard Hexagon Model. III. q-Trinomial Coefficients

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In the first two papers in this series we considered an extension of the hard hexagon model to a solvable two-dimensional lattice gas with at most two particles per pair of adjacent sites, and we described the local densities in terms of elliptic theta functions. Here we present the mathematical theory behind our derivation of the local densities. Our work centers on q-analogs of trinomial coefficients.

KEY WORDS: Statistical mechanics; lattice statistics; number theory; hard hexagon model; Rogers–Ramanujan identities; trinomial coefficients; q-series.

1. INTRODUCTION

This is the third paper in our series[1,2] on lattice gas generalizations of the hard hexagon model. In the first paper we noted that it might be possible to find solvable square-lattice statistical mechanics models corresponding to Gordon's[3,4] generalization of the Rogers–Ramanujan identities. This idea was strongly suggested by the intimate connection between the Rogers–Ramanujan identities[5,6] and the original solution of the hard hexagon model.[7]

In the first paper[1] (subsequently referred to as I) we obtained the solution of the star–triangle relation in the case of two particles per site and we found the local densities for each of the four regimes as multiple sums. We note that Kuniba *et al.* independently obtained this solution of the star–triangle relation, and that it has now been further generalized,[8–12] in particular, to an arbitrary number of particles per site. In the second

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paper[2] (subsequently called II), we presented the local densities for the four regimes in terms of elliptic theta functions and deduced the critical behavior. Here we give the mathematical derivation of the results in II.

Surprisingly, the extensive literature on q-series and elliptic theta functions failed to provide the mathematical tools necessary for our derivations. Briefly, what is needed is a q-analog of trinomial coefficients, i.e., the coefficients of x^j in $(1+x+x^2)^n$. The literature is sparse on trinomial coefficients perhaps because they lack both depth and elegance. It was consequently a great surprise to us that q-analogs of trinomial coefficients were both the key to the mathematics of our model and fairly complicated mathematical objects.

Very recently in a brilliant tour de force, Date *et al.*[21] have generalized the problem yet further, to a double hierarchy of models specified by two-integer L and N, where $L-3 \geqslant N \geqslant 0$. They have shown that the star-triangle relation is satisfied, and derived (by a method different from ours) the local densities.

2. q-TRINOMIAL COEFFICIENTS

2.1. Fundamentals

We begin by considering the polynomial $(1+x+x^2)^n$. The coefficients of the expanded form of this expression are called trinomial coefficients.[13] We shall not follow Comtet's notation exactly, since a variation seems more natural for our purposes:

$$(1+x+x^2)^n = \sum_{j=-n}^{n} \binom{n}{j}_2 x^{j+n} \tag{2.1}$$

By double applications of the binomial theorem to the second and third expressions in

$$(1+x+x^2)^n = [1+x(1+x)]^n = [(1+x)^2 - x]^n \tag{2.2}$$

we find by coefficient comparison that

$$\binom{n}{j}_2 = \sum_{h \geqslant 0} \frac{n!}{h!\,(h+j)!\,(n-j-2h)!} \tag{2.3}$$

$$= \sum_{h \geqslant 0} (-1)^h \binom{n}{h} \binom{2n-2h}{n-j-h} \tag{2.4}$$

Furthermore, we easily deduce from (2.1) that

$$\binom{n}{j}_2 = \binom{n}{-j}_2 \tag{2.5}$$

and

$$\binom{n}{j}_2 = \binom{n-1}{j-1}_2 + \binom{n-1}{j}_2 + \binom{n-1}{j+1}_2 \tag{2.6}$$

Equations (2.3) and (2.4) provide two representations of $\binom{n}{j}_2$, and these lead to six apparently distinct q-analogs:

$$\binom{m;\,B;\,q}{A}_2 = \sum_{j\geqslant 0} \frac{q^{j(j+B)}(q)_m}{(q)_j(q)_{j+A}(q)_{m-2j-A}} \tag{2.7}$$

$$T_0(m, A, q) = \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix} \tag{2.8}$$

$$T_1(m, A, q) = \sum_{j=0} (-q)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix} \tag{2.9}$$

$$t_0(m, A, q) = \sum_{j=0}^{m} (-1)^j q^{j^2} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix} \tag{2.10}$$

$$t_1(m, A, q) = \sum_{j=0}^{m} (-1)^j q^{j^2-j} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix} \tag{2.11}$$

$$\tau_0(m, A, q) = \sum_{j=0}^{m} (-1)^j q^{mj-\binom{j}{2}} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix} \tag{2.12}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix}_q = \begin{bmatrix} A \\ B \end{bmatrix} = \prod_{j=1}^{B} \frac{(1-q^{A+1-j})}{(1-q^j)} \tag{2.13}$$

$$(A;\,q)_n = (A)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}) \tag{2.14}$$

If $q=1$, then each of the above polynomials is equal to $\binom{n}{j}_2$ by (2.3) or (2.4). We next establish the analogs of (2.5) and (2.6) that will be important to us.

First we note that each of (2.8)–(2.12) is symmetric in A and $-A$. As for (2.7), we note

$$\begin{aligned} \binom{m;\,B;\,q}{-A}_2 &= \sum_{j\geqslant 0} \frac{q^{j(j+B)}(q)_m}{(q)_j(q)_{j-A}(q)_{m-2j+A}} \\ &= \sum_{j\geqslant 0} \frac{q^{(j+A)(j+A+B)}(q)_m}{(q)_{j+A}(q)_j(q)_{m-2j-A}} \\ &= q^{A(A+B)} \binom{m;\,B+2A;\,q}{A}_2 \end{aligned} \tag{2.15}$$

2.2. Recurrences

There are numerous analogs of (2.6). We present several. First,

$$T_1(m, A, q) = T_1(m-1, A, q) + q^{m+A} T_0(m-1, A+1, q) + q^{m-A} T_0(m-1, A-1, q) \tag{2.16}$$

The proof of (2.1) is somewhat intricate:

$$\begin{aligned} &T_1(m-1, A, q) + q^{m+A} T_0(m-1, A+1, q) \\ &\quad = \sum_{j\geqslant 0} (-q)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2} \left(\begin{bmatrix} 2m-2j-2 \\ m+A-j-1 \end{bmatrix} + q^{m+A-j} \begin{bmatrix} 2m-2j-2 \\ m+A-j \end{bmatrix} \right) \\ &\quad = \sum_{j\geqslant 0} (-q)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j-1 \\ m+A-j \end{bmatrix} \end{aligned} \tag{2.17}$$

[by Ref. 4, p. 35, Eq. (3.3.4)]

Hence

$$\begin{aligned} &T_1(m, A, q) - T_1(m-1, A, q) - q^{m+A} T_0(m-1, A+1, q) \\ &\quad = \sum_{j\geqslant 0} (-q)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m+A-j \end{bmatrix} \\ &\quad\quad - \sum_{j\geqslant 0} (-q)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j-1 \\ m+A-j \end{bmatrix} \\ &\quad = \sum_{j\geqslant 0} (-1)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2} \frac{(q)_{2m-2j-1} [(1-q^{2m}) - (1-q^{m-A-j})]}{(q)_{m+A-j} (q)_{m-A-j}} \\ &\quad = \sum_{j\geqslant 0} (-q)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2} \frac{(q)_{2m-2j-1} q^{m-A-j} [1-q^{m+A+j}]}{(q)_{m+A-j} (q)_{m-A-j}} \\ &\quad = q^{m-A} \sum_{j\geqslant 0} (-1)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2} \\ &\quad\quad \times \frac{(q)_{2m-2j-1} [(1-q^{m+A-j}) + q^{m+A-j}(1-q^{2j})]}{(q)_{m+A-j} (q)_{m-A-j}} \\ &\quad = q^{m-A} \left\{ \sum_{j\geqslant 0} (-1)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j-1 \\ m+A-j-1 \end{bmatrix} \right. \\ &\quad\quad \left. + \sum_{j\geqslant 0} (-1)^j \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m+A-j \end{bmatrix} \right\} \end{aligned}$$

$$= q^{m-A}\left\{\sum_{j\geqslant 0}(-1)^j\begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2}\begin{bmatrix} 2m-2j-1 \\ m+A-j-1 \end{bmatrix}\right.$$

$$\left.-\sum_{j\geqslant 0}(-1)^j q^{m+A-j-1}\begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2}\begin{bmatrix} 2m-2j-2 \\ m+A-j-1 \end{bmatrix}\right\}$$

$$= q^{m-A}\sum_{j\geqslant 0}(-1)^j\begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2}\begin{bmatrix} 2m-2j-2 \\ m+A-j-2 \end{bmatrix}$$

$$= q^{m-A}T_0(m-1, A-1, q) \qquad \text{[by Ref. 4, p. 35, Eq. (3.3.4)]} \tag{2.18}$$

as desired.

A second q-analog of (2.6) is

$$T_0(m, A, q) = T_0(m-1, A-1, q) + q^{m+A}T_1(m-1, A, q) + q^{2m+2A}T_0(m-1, A+1, q) \tag{2.19}$$

The proof of (2.19) relies on the proof of (2.16); namely, by (2.17)

$$T_0(m, A, q) - q^{m+A}[T_1(m-1, A, q) + q^{m+A}T(m-1, A+1, q)]$$

$$= \sum_{j\geqslant 0}(-1)^j\begin{bmatrix} m \\ j \end{bmatrix}_{q^2}\begin{bmatrix} 2m-2j \\ m+A-j \end{bmatrix}$$

$$-\sum_{j\geqslant 0}(-1)^j q^{m+A+j}\begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2}\begin{bmatrix} 2m-2j-1 \\ m+A-j \end{bmatrix}$$

$$= \sum_{j\geqslant 0}(-1)^j\begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2}\frac{(q)_{2m-2j-1}[(1-q^{2m}) - q^{m+A+j}(1-q^{m-A-j})]}{(q)_{m+A-j}(q)_{m-A-j}}$$

$$= \sum_{j\geqslant 0}(-1)^j\begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2}\frac{(q)_{2m-2j-1}(1-q^{m+A+j})}{(q)_{m+A-j}(q)_{m-A-j}}$$

$$= q^{-m+A}[q^{m-A}T_0(m-1, A-1, q)]$$

[by the last six equations in (2.18)]

Next we consider some recurrences that reduce to tautologies when $q=1$. First,

$$T_1(m, A, q) - q^{m-A}T_0(m, A, q) - T_1(m, A+1, q) + q^{m+A+1}T_0(m, A+1, q) = 0 \tag{2.20}$$

We prove this in two steps:

$$T_1(m, A, q) - q^{m-A} T_0(m, A, q)$$
$$= \sum_{j \geq 0} (-q)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix} (1 - q^{m-A-j})$$
$$= (1-q^{2m}) \sum_{j \geq 0} (-q)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j-1 \\ m-A-j-1 \end{bmatrix}$$
$$= (1-q^{2m}) \sum_{j \geq 0} (-q)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j-1 \\ m+A-j \end{bmatrix} \tag{2.21}$$

Also,

$$T_1(m, A+1, q) - q^{m+A+1} T_0(m, A+1, q)$$
$$= \sum_{j \geq 0} (-q)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m+A+1-j \end{bmatrix} (1 - q^{m+A+1-j})$$
$$= (1-q^{2m}) \sum_{j \geq 0} (-q)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j-1 \\ m+A-j \end{bmatrix} \tag{2.22}$$

Equation (2.20) now follows by substracting (2.22) from (2.21).

Next we have two identities for $\binom{n;B;q}{A}_2$:

$$\binom{m;\, B-1;\, q}{A}_2 = \binom{m;\, B;\, q}{A}_2 + q^B(1-q^m) \binom{m-1;\, B+1;\, q}{A+1}_2 \tag{2.23}$$

$$\binom{m;\, B;\, q}{A}_2 = q^{m-A} \binom{m;\, B-2;\, q}{A}_2 + (1-q^m) \binom{m-1;\, B;\, q}{A}_2 \tag{2.24}$$

Each is easily proved. First (2.23):

$$\binom{m;\, B-1;\, q}{A}_2 - \binom{m;\, B;\, q}{A}_2$$
$$= \sum_{j \geq 0} \frac{q^{j(j+B-1)} (q)_m (1-q^j)}{(q)_j\, (q)_{j+A} (q)_{m-2j-A}}$$
$$= \sum_{j \geq 0} \frac{q^{(j+1)(j+B)} (q)_m}{(q)_j\, (q)_{j+A+1} (q)_{m-2j-A-2}}$$
$$= q^B (1-q^m) \binom{m-1;\, B+1;\, q}{A+1}_2$$

Now, by (2.24)

$$\begin{aligned}
&\binom{m;\, B;\, q}{A}_2 - q^{m-A}\binom{m;\, B-2;\, q}{A}_2 \\
&= \sum_{j \geqslant 0} \frac{q^{j(j+B)}(q)_m(1-q^{m-2j-A})}{(q)_j\,(q)_{j+A}\,(q)_{m-2j-A}} \\
&= (1-q^m) \sum_{j \geqslant 0} \frac{q^{j(j+B)}\,(q)_{m-1}}{(q)_j\,(q)_{j+A}\,(q)_{m-2j-A-1}} \\
&= (1-q^m)\binom{m-1;\, B;\, q}{A}_2
\end{aligned}$$

We close this subsection with restatements of (2.16), (2.19), and (2.20) for the $\binom{m;B;q}{A}_2$ and we add two further recurrences:

$$\binom{m;\, A-1;\, q}{A}_2 = q^{m-1}\binom{m-1;\, A-1;\, q}{A}_2 + q^A\binom{m-1;\, A+1;\, q}{A+1}_2 + \binom{m-1;\, A-1;\, q}{A-1}_2 \tag{2.25}$$

$$\binom{m;\, A;\, q}{A}_2 = q^{m-A}\binom{m-1;\, A-1;\, q}{A-1}_2 + q^{m-A-1}\binom{m-1;\, A-1;\, q}{A}_2 + \binom{m-1;\, A+1;\, q}{A+1}_2 \tag{2.26}$$

$$\binom{m;\, A;\, q}{A}_2 + q^m\binom{m;\, A;\, q}{A+1}_2 - \binom{m;\, A+1;\, q}{A+1}_2 - q^{m-A}\binom{m-1;\, A-1;\, q}{A+1}_2 = 0 \tag{2.27}$$

$$\binom{m;\, B;\, q}{A}_2 = \binom{m-1;\, B;\, q}{A}_2 + q^{m-A-1+B}\binom{m-1;\, B;\, q}{A+1}_2 + q^{m-A}\binom{m-1;\, B-1;\, q}{A-1}_2 \tag{2.28}$$

$$\binom{m;\, B;\, q}{A}_2 = \binom{m-1;\, B;\, q}{A}_2 + q^{m-A}\binom{m-1;\, B-2;\, q}{A-1}_2 + q^{m+B}\binom{m-1;\, B+1;\, q}{A+1}_2 \tag{2.29}$$

Now (2.29) is merely (2.28) with A replaced by $-A$, B by $B-2A$, and (2.15) applied. As for (2.28), we see that

$$\begin{pmatrix} m; B; q \\ A \end{pmatrix}_2 - \begin{pmatrix} m-1; B; q \\ A \end{pmatrix}_2$$

$$= \sum_{j \geqslant 0} \frac{q^{j(j+B)}(q)_{m-1} [(1-q^m)-(1-q^{m-2j-A})]}{(q)_j (q)_{j+A} (q)_{m-2j-A}}$$

$$= \sum_{j \geqslant 0} \frac{q^{j(j+B)}(q)_{m-1} q^{m-2j-A}[(1-q^j)+q^j(1-q^{j+A})]}{(q)_j (q)_{j+A} (q)_{m-2j-A}}$$

$$= \sum_{j \geqslant 0} \frac{q^{(j+1)(j+1+B)+m-2(j+1)-A}(q)_{m-1}}{(q)_j (q)_{j+A+1} (q)_{m-1-2j-(A+1)}}$$

$$+ \sum_{j \geqslant 0} \frac{q^{j(j+B-1)+m-A}(q)_{m-1}}{(q)_j (q)_{j+A-1} (q)_{m-1-2j-(A-1)}}$$

$$= q^{m-A-1+B} \begin{pmatrix} m-1; B; q \\ A+1 \end{pmatrix}_2 + q^{m-A} \begin{pmatrix} m-1; B-1; q \\ A-1 \end{pmatrix}_2$$

which proves (2.28).

2.3. Generating Functions

The generating function for $\binom{n}{j}_2$ is quite simple and is given by (2.1). The related generating functions for the q-analogs are not so simple; however, they prove useful in applications.

Recall the q-hypergeometric function (Ref. 14, p. 65) given by

$$_r\phi_s \begin{pmatrix} a_1, a_2, \ldots, a_r; q, t \\ b_1, \ldots, b_s \end{pmatrix} = \sum_{n \geqslant 0} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n t^n}{(q)_n (b_1)_n (b_2)_n \cdots (b_s)_n} \tag{2.30}$$

Then

$$\sum_{A=-n}^{n} x^{n+A} q^{\binom{A}{2}} \begin{pmatrix} n; A-r; q \\ A \end{pmatrix}_2 = q^{\binom{n+1}{2}-nr} {}_2\phi_1 \begin{pmatrix} q^{-n}, -x; q, -xq^r \\ 0 \end{pmatrix} \tag{2.31}$$

To see this, we observe that

$$\sum_{A=-n}^{n} x^{n+A} q^{\binom{A}{2}} \begin{pmatrix} n; A-r; q \\ A \end{pmatrix}_2$$

$$= \sum_{j \geqslant 0} q^{j(j-r)} \begin{bmatrix} n \\ j \end{bmatrix} x^n \sum_{A=-n}^{n} x^A q^{\binom{A}{2}+Aj} \begin{bmatrix} n-j \\ j+A \end{bmatrix}$$

[by (2.7)]

$$= \sum_{j \geq 0} q^{j(j-r)} \begin{bmatrix} n \\ j \end{bmatrix} x^n \sum_{A=-\infty}^{\infty} x^{A-j} q^{\binom{A-j}{2}+(A-j)j} \begin{bmatrix} n-j \\ A \end{bmatrix}$$

$$= \sum_{j \geq 0} q^{\binom{j+1}{2}-jr} \begin{bmatrix} n \\ j \end{bmatrix} x^{n-j}(-x)_{n-j}$$

[by Ref. 4, Theorem 3.3, p. 36]

$$= \sum_{j \geq 0} q^{\binom{n-j+1}{2}-(n-j)r} \begin{bmatrix} n \\ j \end{bmatrix} x^j(-x)_j$$

$$= q^{\binom{n+1}{2}-nr} {}_2\phi_1 \begin{pmatrix} q^{-n}, -x; q, -xq^r \\ 0 \end{pmatrix}$$

We also have

$$\sum_{A=-n}^{n} x^{n+A} q^{\binom{A}{2}} \tau_0(n, A, q)$$

$$= q^{\binom{n+1}{2}} \sum_{A=-n}^{n} x^{n+A} q^{\binom{A}{2}} \sum_{j \geq 0} (-1)^{n-j} q^{-\binom{-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} 2j \\ A+j \end{bmatrix}$$

$$= q^{\binom{n+1}{2}} \sum_{j \geq 0} (-1)^{n-j} q^{-\binom{j+1}{2}} \begin{bmatrix} n \\ j \end{bmatrix} x^n \sum_{A=-\infty}^{\infty} x^{A-j} q^{\binom{A-j}{2}} \begin{bmatrix} 2j \\ A \end{bmatrix}$$

$$= q^{\binom{n+1}{2}} \sum_{j \geq 0} (-1)^{n-j} x^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} (-xq^{-j})_{2j}$$

$$= (-x)^n q^{\binom{n+1}{2}} \sum_{j \geq 0} \frac{(q^{-n})_j (-x)_j (-x^{-1}q)_j q^{j(n-j)}}{(q)_j}$$

$$= (-x)^n q^{\binom{n+1}{2}} \lim_{\tau \to 0} {}_3\phi_2 \begin{pmatrix} q^{-n}, -x, -x^{-1}q; q, \tau^{-2}q^n \\ \tau^{-1}, \tau^{-1}q \end{pmatrix} \qquad (2.32)$$

2.4. Identities

There are really only two families of polynomials of interest in Eqs. (2.7)–(2.12). The apparently different polynomials listed there are related by the following identities:

$$T_i(m, A, q^{-1}) = q^{A^2-m^2} t_i(m, A, q), \qquad i = 0, 1 \qquad (2.33_i)$$

$$\begin{pmatrix} m; A-i; q^2 \\ A \end{pmatrix}_2 = q^{i(A-m)} t_i(m, A, q), \qquad i = 0, 1 \qquad (2.34_i)$$

$$\tau_0(m, A, q) = \begin{pmatrix} m; A; q \\ A \end{pmatrix}_2 \qquad (2.35)$$

Identity (2.33) is quite easy. One merely replaces q by q^{-1} in (2.8) and (2.9) and then simplifies by using the trivial identity

$$\begin{bmatrix} A \\ B \end{bmatrix}_{q^{-1}} = q^{-B(A-B)} \begin{bmatrix} A \\ B \end{bmatrix} \tag{2.36}$$

To prove (2.34) it is necessary to prove a third identity also. Define

$$t_2(m, A, q) = \sum_{j \geqslant 0} (-1)^j q^{j^2 - j} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-j-A \end{bmatrix} \times \frac{(1 + q^{2A} - q^{m+A+j-1} - q^{m+A-j})}{(1 - q^{2m-2j-1})} \tag{2.37}$$

We shall prove (2.34) simultaneously with

$$\begin{pmatrix} m;\ A-2;\ q^2 \\ A \end{pmatrix}_2 = q^{A-m} t_2(m, A, q) \tag{2.38}$$

We note directly from the relevant definition that

$$\begin{pmatrix} m;\ B;\ q^2 \\ A \end{pmatrix}_2 = t_0(m, m, q) = t_1(m, m, q) = t_2(m, m, q) = 1$$

and

$$\begin{pmatrix} m;\ -m;\ q^2 \\ -m \end{pmatrix}_2 = t_0(m, -m, q) = 1$$

$$\begin{pmatrix} m;\ -m-1;\ q^2 \\ -m \end{pmatrix} = q^{-2m} = q^{(-m)-m} t_1(m, -m, q)$$

$$\begin{pmatrix} m;\ -m-2;\ q^2 \\ -m \end{pmatrix}_2 = q^{-4m} = q^{-2m} \frac{(1 + q^{-2m} - q^{-1} - 1)}{(1 - q^{2m-1})} = q^{(-m)-m} t_2(m, -m, q)$$

Hence (2.34_0), (2.34_1), and (2.38) are valid for $A = \pm m$, and for $|A| > m$ they are the tautology "$0 = 0$."

We proceed by induction on m. If $m = 0$, then either $A = 0$ or $|A| > 0$ and both these cases have been established above. By (2.23) with $B = A$,

$$\begin{pmatrix} m;\ A-1;\ q^2 \\ A \end{pmatrix}_2 = \begin{pmatrix} m;\ A;\ q^2 \\ A \end{pmatrix}_2 + q^{2A}(1 - q^{2m}) \begin{pmatrix} m-1;\ A+1;\ q^2 \\ A+1 \end{pmatrix}_2 \tag{2.39}$$

By (2.23) with $B=A-1$

$$\binom{m;\, A-2;\, q^2}{A}_2 = \binom{m;\, A-1;\, q^2}{A}_2 + q^{2(A-1)}(1-q^{2m})\binom{m-1;\, A;\, q^2}{A+1}_2 \tag{2.40}$$

And by (2.24) with $B=A$

$$\binom{m;\, A;\, q^2}{A}_2 = q^{2(m-A)}\binom{m;\, A-2;\, q^2}{A}_2 + (1-q^{2m})\binom{m-1;\, A;\, q^2}{A}_2 \tag{2.41}$$

Now Eqs. (2.39)–(2.41) can be viewed as recurrences that relate

$$\binom{m;\, A-i;\, q^2}{A}_2 \qquad (i=0,\, 1,\, 2)$$

to the same q-trinomial coefficients with m replaced by $m-1$. Furthermore, if we regard (2.39)–(2.41) as a linear system in

$$\binom{m;\, A-i;\, q^2}{A}_2 \qquad (i=0,\, 1,\, 2)$$

then the determinant of the system is

$$\begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -q^{2m-2A} \end{vmatrix} = 1-q^{2m-2A} \tag{2.42}$$

and this expression is nonzero unless $A=m$; however, we have already disposed of the case $A=m$. Therefore, if we can establish that the right-hand sides of (2.34_0), (2.34_1), and (2.38) satisfy (2.39)–(2.41), then we will have proved the necessary and sufficient recurrences for the induction step passing from $m-1$ to m. We have

$$\begin{aligned} &q^{A-m}t_1(m, A, q) - t_0(m, A, q) \\ &\quad = q^{A-m} \sum_{j\geqslant 0} (-1)^j q^{j^2-j} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \\ &\qquad \times \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix} [(1-q^{2j}) + q^{2j}(1-q^{m-A-j})] \\ &\quad = q^{A-m}(1-q^{2m}) \sum_{j\geqslant 0} (-1)^j q^{j^2-j} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-j-A \end{bmatrix} \\ &\qquad + q^{A-m}(1-q^{2m}) \sum_{j\geqslant 0} (-1)^j q^{j^2+j} \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j-1 \\ m-j-A-1 \end{bmatrix} \end{aligned}$$

$$= q^{A-m}(1-q^{2m})\left(-\sum_{j\geq 0}(-1)^j q^{j^2+j}\begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2}\begin{bmatrix} 2m-2j-1 \\ m-j-A-1 \end{bmatrix}\right.$$

$$\left.+\sum_{j\geq 0}(-1)^j q^{j^2+j}\begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2}\begin{bmatrix} 2m-2j-1 \\ m-j-A-1 \end{bmatrix}\right)$$

$$= q^{A-m}(1-q^{2m})\sum_{j\geq 0}(-1)^j q^{j^2+j}\begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2}\begin{bmatrix} 2m-2j-2 \\ m-j-A-2 \end{bmatrix} q^{m-j+A}$$

[by Ref. 4, p. 35, Eq. (3.3.3)]

$$= q^{2A}(1-q^{2m})\, t_0(m-1, A+1, q) \tag{2.43}$$

which is (2.39)

Next we treat (2.41):

$$t_0(m, A, q) - (1-q^{2m})\, t_0(m-1, A, q)$$

$$= \sum_{j\geq 0}(-1)^j q^{j^2}\begin{bmatrix} m \\ j \end{bmatrix}_{q^2}\begin{bmatrix} 2m-2j \\ m-j-A \end{bmatrix}$$

$$-\sum_{j\geq 0}(-1)^j q^{j^2}\begin{bmatrix} m \\ j \end{bmatrix}_{q^2}(1-q^{2m-2j})\begin{bmatrix} 2m-2j-2 \\ m-j-A-1 \end{bmatrix}$$

$$= \sum_{j\geq 0}(-1)^j q^{j^2}\begin{bmatrix} m \\ j \end{bmatrix}_{q^2}\frac{(q)_{2m-2j-2}(1-q^{2m-2j})}{(q)_{m-j-A}\,(q)_{m-j+A}}$$

$$\times[(1-q^{2m-2j-1})-(1-q^{m-j-A})(1-q^{m-j+A})]$$

$$= q^{m-A}\sum_{j\geq 0}(-1)^j q^{j^2-j}\begin{bmatrix} m \\ j \end{bmatrix}_{q^2}$$

$$\times\begin{bmatrix} 2m-2j \\ m-j-A \end{bmatrix}\frac{(1+q^{2A}-q^{m+A-j-1}-q^{m+A-j})}{(1-q^{2m-2j-1})}$$

$$= q^{2m-2A}[q^{A-m}t_2(N, A, q)] \tag{2.44}$$

which is (2.41).

Finally, we consider (2.40):

$$q^{A-m}t_1(m, A, q) - q^{A-m}t_2(m, A, q)$$

$$= q^{A-m}\left\{\sum_{j\geq 0}(-1)^j q^{j^2-j}\begin{bmatrix} m \\ j \end{bmatrix}_{q^2}\begin{bmatrix} 2m-2j \\ m-j-A \end{bmatrix}\right.$$

$$-\sum_{j\geq 0}(-1)^j q^{j^2-j}\begin{bmatrix} m \\ j \end{bmatrix}_{q^2}$$

$$\left.\times\begin{bmatrix} 2m-2j \\ m-j-A \end{bmatrix}\frac{(1+q^{2A}-q^{m+A-j-1}-q^{m+A-j})}{(1-q^{2m-2j-1})}\right\}$$

$$= q^{A-m} \sum_{j \geqslant 0} (-1)^j q^{j^2 - j} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-j-A \end{bmatrix} \frac{1}{(1-q^{2m-2j-1})}$$
$$\times (1 - q^{2m-2j-1} - 1 - q^{2A} + q^{m+A-j-1} + q^{m+A-j})$$
$$= q^{A-m} \sum_{j \geqslant 0} (-1)^j q^{j^2-j} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2}$$
$$\times \begin{bmatrix} 2m-2j \\ m-j-A \end{bmatrix} \frac{(1-q^{m-A-j})(-q^{2A}+q^{m+A-j-1})}{(1-q^{2m-2j-1})}$$
$$= -q^{3A-m}(1-q^{2m}) \sum_{j \geqslant 0} (-)^j q^{j^2-j} \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j-2 \\ m-j-A-2 \end{bmatrix}$$
$$= -q^{2(A-1)}(1-q^{2m})\, q^{(A+1)-(m-1)} t_1(m-1, A+1, q) \tag{2.45}$$

which is (2.40).

The necessary recurrences have been established to complete our proof of (2.34_0), (2.34_1), and (2.38) by mathematical induction.

We now turn to (2.35). By (2.31) and (2.32), we see that to establish (2.35) we need only prove

$$q^{-nr}\, {}_2\phi_1 \begin{pmatrix} q^{-n}, -x; q, -xq^r \\ 0 \end{pmatrix}$$
$$= (-x)^n \lim_{\tau \to 0} {}_3\phi_2 \begin{pmatrix} q^{-n}, -x, -x^{-1}q; q, \tau^{-2}q^n \\ \tau^{-1}, \tau^{-1}q \end{pmatrix} \tag{2.46}$$

This is merely a limiting case of [Ref. 15; Eq. (10.2)]

$${}_3\phi_2 \begin{pmatrix} q^{-n}, b, c; q, efq^n/bc \\ e, f \end{pmatrix} = \frac{(e/b)_n}{(e)_n}\, {}_3\phi_2 \begin{pmatrix} q^{-n}, b, f/c; q, q \\ q^{1-n}b/e, f \end{pmatrix} \tag{2.47}$$

by taking $b = -x$, $c = -x^{-1}q$, $e = \tau^{-1}$, $f = \tau^{-1}q$.

2.5. Asymptotics

In subsequent sections it will be very important to know what happens to the q-trinomial coefficients as $m \to \infty$:

$$\lim_{m \to \infty} \begin{pmatrix} m; A; q \\ A \end{pmatrix}_2 = \lim_{m \to \infty} \tau_0(m, A, q) = \frac{1}{(q)_\infty} \tag{2.48}$$

$$\lim_{m \to \infty} \begin{pmatrix} m; A-1; q \\ A \end{pmatrix}_2 = \frac{1+q^A}{(q)_\infty} \tag{2.49}$$

$$\lim_{m\to\infty} t_0(m, A, q) = \frac{1}{(q^2; q^2)_\infty} \tag{2.50}$$

$$\lim_{m\to\infty} T_1(m, A, q) = \frac{(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \tag{2.51}$$

$$\lim_{m\to\infty} q^{-m} t_1(m, A, q) = \frac{q^{-A} + q^A}{(q^2; q^2)_\infty} \tag{2.52}$$

$$\lim_{\substack{m\to\infty \\ m-A \text{ even}}} T_0(m, A, q) = \frac{\alpha(q^2)}{(q^2; q^2)_\infty} \tag{2.53}$$

$$\lim_{\substack{m\to\infty \\ m-A \text{ odd}}} T_0(m, A, q) = \frac{q\beta(q^2)}{(q^2; q^2)_\infty} \tag{2.54}$$

where

$$\begin{aligned}\alpha(q) &= (q; q^8)_\infty (q^7; q^8)_\infty (q^8; q^8)_\infty (q^6; q^{16})_\infty (q^{10}; q^{16})_\infty (q)_\infty^{-1} \\ &= [(-q^{11}: q^{24})_\infty (-q^{13}; q^{24})_\infty (q^{24}; q^{24})_\infty \\ &\quad - q(-q^5; q^{24})_\infty (-q^{19}; q^{24})_\infty (q^{24}; q^{24})_\infty](q)_\infty^{-1} \\ &= \tfrac{1}{2}[(-q^{1/2}; q)_\infty + (q^{1/2}; q)_\infty]\end{aligned} \tag{2.55}$$

and

$$\begin{aligned}\beta(q) &= (q^3; q^8)_\infty (q^5; q^8)_\infty (q^8; q^8)_\infty (q^2; q^{16})_\infty (q^{14}; q^{16})_\infty (q)_\infty^{-1} \\ &= [(-q^7; q^{24})_\infty (-q^{17}; q^{24})_\infty (q^{24}; q^{24})_\infty \\ &\quad - q^2(-q; q^{24})_\infty (-q^{23}; q^{24})_\infty (q^{24}; q^{24})_\infty](q)_\infty^{-1} \\ &= q^{-1/2}[(-q^{1/2}; q)_\infty - (q^{1/2}; q)_\infty]\end{aligned} \tag{2.56}$$

Note that the limits in (2.53) and (2.54) are not taken as m tends to infinity through all integral values, but only those values of m of the indicated parity are allowed.

To obtain (2.48) and (2.49), we note

$$\begin{aligned}\lim_{m\to\infty} \begin{pmatrix} m; A; q \\ A \end{pmatrix}_2 &= \sum_{j\geqslant 0} \frac{q^{j(j+B)}}{(q)_j (q)_{j+A}} \\ &= \frac{1}{(q)_\infty} \sum_{j=0}^{A-B} \begin{bmatrix} A-B \\ j \end{bmatrix} q^{j^2+jB}\end{aligned} \tag{2.57}$$

[by Ref. 14, p. 576, Eq. (12), $\alpha = \beta = 1/t$,

$\tau = t^2 q^{1+B}$, $c = q^{A+1}$, $t \to 0$]

Taking (2.35) into account, we see that (2.48) is the case $A = B$ of (2.57) and (2.49) is the case $A-1=B$ of (2.57).

Next we treat (2.50):

$$\lim_{m\to\infty} t_0(m, A, q) = \lim_{m\to\infty} \sum_{j\geqslant 0} (-1)^j q^{j^2} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix}$$

$$= \frac{1}{(q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j q^{j^2}}{(q^2;q^2)_j} = \frac{(q;q^2)_\infty}{(q)_\infty} = \frac{1}{(q^2;q^2)_\infty}$$

[by Ref. 4, p. 19, Eq. (2.2.6)]

For (2.51) we see that

$$\lim_{m\to\infty} T_1(m, A, q) = \lim_{m\to\infty} \sum_{j=0}^{m} (-q)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix}$$

$$= \frac{1}{(q)_\infty} \sum_{j=0}^{\infty} \frac{(-q)^j}{(q^2;q^2)_j} = \frac{1}{(-q;q^2)_\infty (q)_\infty} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}$$

[by Ref. 4, p. 19, Eq. (2.2.5) and p. 5, Eq. (1.2.5)]

Now, for (2.52)

$$\lim_{m\to\infty} q^{-m} t_1(m, A, q) = q^{-A} \lim_{m\to\infty} \begin{pmatrix} m; A-1; q \\ A \end{pmatrix}_2$$

$$= \frac{q^{-A}(1+q^{2A})}{(q^2;q^2)_\infty}$$

by (2.49).

The limits in (2.53) and (2.54) are the toughest. To obtain them, we need the following identities, which are easily obtained from (2.7) by reversing the index of summation:

If $m-A$ is even,

$$q^{(m-A)(m+B)/2} \begin{pmatrix} m; B; q^{-1} \\ A \end{pmatrix}_2 = \sum_{j\geqslant 0} \frac{q^{2j^2+(B-A)j}(q)_m}{(q)_{2j}\,(q)_{(m-A-2j)/2}\,(q)_{(m+A-2j)/2}} \tag{2.58}$$

If $m-A$ is odd,

$$q^{(m-A)(m+B)/2+(A-B-1)/2} \begin{pmatrix} m; B; q^{-1} \\ A \end{pmatrix}_2$$

$$= \sum_{j\geqslant 0} \frac{q^{2j^2+(B-A+2)j}(q)_m}{(q)_{2j+1}(q)_{(m-A-1-2j)/2}\,(q)_{(m+A-1-2j)/2}} \tag{2.59}$$

We now prove (2.53):

$$\lim_{\substack{m\to\infty\\ m-A\text{ even}}} T_0(m, A, q)$$

$$= \lim_{\substack{m\to\infty\\ m-A\text{ even}}} q^{m^2-A^2} t_0(m, A, q^{-1}) \qquad [\text{by } (2.33_0)]$$

$$= \lim_{\substack{m\to\infty\\ m-A\text{ even}}} q^{m^2-A^2} \binom{m;\, A;\, q^{-2}}{A}_2 \qquad [\text{by } (2.34_0)]$$

$$= \lim_{m\to\infty} \sum_{j\geqslant 0} \frac{q^{4j^2}(q^2;q^2)_m}{(q^2;q^2)_{2j}\,(q^2;q^2)_{(m-A-2j)/2}\,(q^2;q^2)_{(m+A-2j)/2}}$$

[by (2.58)]

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{j=0}^{\infty} \frac{q^{4j^2}}{(q^2;q^2)_{2j}}$$

$$= \frac{\alpha(q^2)}{(q^2;q^2)_\infty} \qquad [\text{by Ref. 17, p. 160, Eq. (83)}] \qquad (2.60)$$

We conclude this section with (2.54):

$$\lim_{\substack{m\to\infty\\ m-A\text{ odd}}} T_0(m, A, q)$$

$$= \lim_{\substack{m\to\infty\\ m-A\text{ odd}}} q^{m^2-A^2} t_0(m, A, q^{-1}) \qquad [\text{by } (2.33_0)]$$

$$= \lim_{\substack{m\to\infty\\ m-A\text{ odd}}} q^{m^2-A^2} \binom{m;\, A;\, q^{-2}}{A}_2 \qquad [\text{by } (2.34_0)]$$

$$= \lim_{m\to\infty} q \sum_{j\geqslant 0} \frac{q^{4j^2+4j}\,(q^2;q^2)_m}{(q^2;q^2)_{2j+1}\,(q^2;q^2)_{(m-A-1-2j)/2}\,(q^2;q^2)_{(m+A-1-2j)/2}}$$

[by (2.59)]

$$= \frac{q}{(q^2;q^2)_\infty} \sum_{j=0}^{\infty} \frac{q^{4j^2+4j}}{(q^2;q^2)_{2j+1}}$$

$$= \frac{q\beta(q^2)}{(q^2;q^2)_\infty} \qquad [\text{by Ref. 17, p. 161, Eq. (86)}] \qquad (2.61)$$

3. REGIME III

In this section we establish the formulas (3.16a)–(3.16d) of II. Our approach is to represent the polynomials $X_m(a, b, c, q^{-1})$ given by (2.6) in II in terms of the q-trinomial coefficients given in Section 2. We begin by defining

$$\begin{aligned} &F_m(j, k, l; q)_\varepsilon \\ &\quad = F_m(j, k, l)_\varepsilon \\ &\quad = \sum_{\mu=-\infty}^{\infty} q^{35\mu^2 + (7k + 7l - 10j - 5)\mu + (j-k)(j-l)} T_\varepsilon(m, 7\mu + k - j, q) \end{aligned} \tag{3.1}$$

where $\varepsilon = 0$ or 1. We then set

$$F_m(j, k, l) = F_m(j, k, l)_{\delta(k,l)} \tag{3.2}$$

where $\delta(k, k) = 1$, $\delta(k, l) = 0$ if $k \neq l$, and we define

$$\rho_0 = 1, \qquad \rho_1 = 3, \qquad \rho_2 = 2 \tag{3.3}$$

Then under conditions (2.5a) and (2.6) of II with $0 \leqslant j, k, l \leqslant 2$:

$$X_m(\rho_j, \rho_k, \rho_l; q^{-1}) = F_m(j, k, l) - q^{-2j-1} F_m(-j-1, k, l) \tag{3.4}$$

To prove (3.4), we need only establish that the right-hand side satisfies the following defining recurrences and initial conditions:

$$X_0(a, b, c; q^{-1}) = \begin{cases} 1 & \text{for } a = b \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

$$X_m(a, b, c; q^{-1}) = \sum_{h=4-b}^{3} q^{|\rho^{-1}(h) - \rho^{-1}(c)|m} X_{m-1}(a, h, b; q^{-1}) \tag{3.6}$$

where, by (3.3), $\rho^{-1}(1) = 0$, $\rho^{-1}(3) = 1$, $\rho^{-1}(2) = 2$, and throughout $b + c \geqslant 4$, $1 \leqslant a, b, c \leqslant 3$.

We remark that (3.6) has been apparently altered from the natural recurrences arising from (2.6) of II; however, inspection shows that it yields exactly the same recurrences that the function $H(a, b, c)$ yields. Furthermore, the ρ-function seems to be the natural one to use in expressing our regime III polynomials succinctly.

To establish (3.5), we note that by (2.8) and (2.9),

$$F_0(j, k, l) = \begin{cases} 1 & \text{if } j = k \\ 0 & j \neq k \end{cases} \tag{3.7}$$

and this shows that the right-hand side of (3.4) satisfies (3.5).

We now translate term by term the identities (2.16), (2.19), and (2.20) into recurrences for the $F_m(j,k,l)_\varepsilon$:

$$\begin{aligned} F_m(j,k,l)_1 &\\ &= F_{m-1}(j,k,l)_1 \\ &\quad + q^{m+k-l}F_{m-1}(j,k+1,l)_0 + q^{m+l-k}F_{m-1}(j,k-1,l)_0 \end{aligned} \tag{3.8}$$

$$\begin{aligned} F_m(j,k,l)_0 &\\ &= q^m F_{m-1}(j,k,l+1) \\ &\quad + q^{2m+k-l-1}F_{m-1}(j,k+1,l+1)_0 \\ &\quad + q^{l-k+1}F_{m-1}(j,k-1,l+1)_0 \end{aligned} \tag{3.9a}$$

$$\begin{aligned} F_m(j,k,l)_0 &\\ &= q^{k-l+1}F_{m-1}(j,k+1,l-1)_0 + q^m F_{m-1}(j,k,l-1)_1 \\ &\quad + q^{2m+l-k-1}F_{m-1}(j,k-1,l-1)_0 \\ &\qquad [\text{by (2.19) with } -A \text{ replacing } A] \end{aligned} \tag{3.9b}$$

$$\begin{aligned} &F_m(j,k,l)_1 - q^m F_m(j,k,l-1)_0 \\ &\quad q^{k-l+1}F_m(j,k+1,l-1)_1 + q^{m+k-l+1}F_m(j,k+1,l)_0 = 0 \end{aligned} \tag{3.10a}$$

$$\begin{aligned} &F_m(j,k,l)_1 - q^m F_m(j,k,l+1)_0 - q^{l+1-k}F_m(j,k-1,l+1)_1 \\ &\quad q^{m+l+1-k}F_m(j,k-1,l)_0 = 0 \\ &\quad [\text{by (2.20) with } -A \text{ replacing } A] \end{aligned} \tag{3.10b}$$

Next we consider (3.1) with μ replaced by $-\mu$:

$$F_m(j,k,l)_\varepsilon = q^{2k-2j}F_m(-j-1,\ -k-1,\ -l+1)_\varepsilon \tag{3.11}$$

Finally we take μ into $\mu+1$ in (3.1):

$$F_m(j,k,l)_\varepsilon = q^{4k+9}F_m(j,k+7,l+3)_\varepsilon \tag{3.12}$$

There are six pairs b, c that are admissible in (3.6). We shall use (3.8)–(3.12) to show that the right-hand side of (3.4) also satisfies (3.6) in these instances. Let $X_m^*(j,k,l;q^{-1})$ denote the right-hand side of (3.4). For $b=c=3$, we have

$$\begin{aligned}
&X_m^*(\rho_j, 3, 3; q^{-1})\\
&\quad = F_m(j, 1, 1) - q^{-2j-1}F_m(-j-1, 1, 1)\\
&\quad = [F_{m-1}(j, 1, 1) + q^m F_{m-1}(j, 2, 1) + q^m F_{m-1}(j, 0, 1)]\\
&\qquad - q^{-2j-1}[F_{m-1}(-j-1, 1, 1) + q^m F_{m-1}(-j-1, 2, 1)\\
&\qquad + q^m F_{m-1}(-j-1, 0, 1)] \qquad [\text{by } (3.8)]\\
&\quad = X_{m-1}^*(\rho_j, 3, 3; q^{-1}) + q^m X_{m-1}^*(j, 2, 3; q^{-1}) + q^m X_{m-1}^*(j, 1, 3; q^{-1})
\end{aligned}$$

which is (3.6).

Next $b=3$, $c=2$:

$$\begin{aligned}
&X_m^*(\rho_j, 3, 2; q^{-1})\\
&\quad = F_m(j, 1, 2) - q^{-2j-1}F_m(-j-1, 1, 2)\\
&\quad = [F_{m-1}(j, 2, 1) + q^m F_{m-1}(j, 1, 1) + q^{2m} F_{m-1}(j, 0, 1)]\\
&\qquad - q^{-2j-1}[F_{m-1}(-j-1, 2, 1) + q^m F_{m-1}(-j-1, 1, 1)\\
&\qquad + q^{2m} F_{m-1}(-j-1, 0, 1)] \qquad [\text{by } (3.9_\mathrm{b})]\\
&\quad = X_m^*(\rho_j, 2, 3; q^{-1}) + q^m X_{m-1}^*(\rho_j, 3, 3; q^{-1}) + q^{2m} X_{m-1}^*(\rho_j, 1, 3; q^{-1})
\end{aligned}$$

For $b=3$, $c=1$:

$$\begin{aligned}
&X_m^*(\rho_j, 3, 1, q^{-1})\\
&\quad = F_m(j, 1, 0) - q^{-2j-1}F_m(-j-1, 1, 0)\\
&\quad = [q^m F_{m-1}(j, 1, 1) + q^{2m} F_{m-1}(j, 2, 1) + F_{m-1}(j, 0, 1)]\\
&\qquad - q^{-2j-1}[q^m F_{m-1}(-j-1, 1, 1) + q^{2m} F_{m-1}(-j-1, 2, 1)\\
&\qquad + F_{m-1}(-j-1, 0, 1)] \qquad [\text{by } (3.9_\mathrm{a})]\\
&\quad = q^m X_{m-1}^*(\rho_j, 3, 3; q^{-1}) + q^{2m} X_{m-1}^*(\rho_j, 2, 3; q^{-1})\\
&\qquad + X_{m-1}^*(\rho_j, 1, 3; q^{-1})
\end{aligned}$$

For $b=2$, $c=3$:

$$\begin{aligned}
&X_m^*(\rho_j, 2, 3; q^{-1})\\
&\quad = F_m(j, 2, 1) - q^{-2j-1}F_m(-j-1, 2, 1)\\
&\quad = [q^m F_{m-1}(j, 2, 2) + q^{2m} F_{m-1}(j, 3, 2) + F_{m-1}(j, 1, 2)]\\
&\qquad - q^{-2j-1}[q^m F_{m-1}(-j-1, 2, 2) + q^{2m} F_{m-1}(-j-1, 3, 2)\\
&\qquad + F_{m-1}(-j-1, 1, 2)] \qquad [\text{by } (3.9_\mathrm{a})]
\end{aligned}$$

$$= q^m X_m^*(\rho_j, 2, 2; q) + X_m^*(\rho_j, 3, 2; q^{-1})$$
$$+ q^{2m}[F_{m-1}(j, 3, 2)_0 - q^{-2j-1}F_{m-1}(-j-1, 3, 2)_0]$$
$$= q^m X_m^*(\rho_j, 2, 2; q^{-1}) + X_m^*(\rho_j, 3, 2; q^{-1})$$
$$+ q^{2m}[q^{6-2j}F_{m-1}(-j-1, -4, -1)_0$$
$$- q^{-2j-1}F_{m-1}(-j-1, 3, 2)_0] \qquad [\text{by } (3.11)]$$
$$= q^m X_m^*(\rho_j, 2, 2; q^{-1}) + X_m^*(\rho_j, 3, 2; q^{-1}) \qquad [\text{by } (3.12)]$$

For $b = 2$, $c = 2$:

$$X_m^*(\rho_j, 2, 2; q^{-1})$$
$$= F_m(j, 2, 2) - q^{-2j-1}F_m(-j-1, 2, 2)$$
$$= [F_{m-1}(j, 2, 2)_1 + q^m F_{m-1}(j, 3, 2)_0 + q^m F_{m-1}(j, 1, 2)_0]$$
$$- q^{-2j-1}[F_{m-1}(-j-1, 2, 2)_1 + q^m F_{m-1}(-j-1, 3, 2)_0$$
$$+ q^m F_{m-1}(-j-1, 1, 2)_0]$$
$$= X_{m-1}^*(\rho_j, 1, 1; q^{-1}) + q^m X_{m-1}^*(\rho_j, 1, 2; q^{-1})$$
$$[\text{as in the case } b = 2, c = 3]$$

For $b = 1$, $c = 3$:

$$X_m^*(\rho_j, 1, 3; q^{-1})$$
$$= F_m(j, 0, 1) - q^{-2j-1}F_m(-j-1, 0, 1)$$
$$= [F_{m-1}(j, 1, 0)_0 + q^m F_{m-1}(j, 0, 0)_1 + q^{2m}F_{m-1}(j, -1, 0)_0]$$
$$- q^{-2j-1}[F_{m-1}(-j-1, 1, 0)_0 + q^m F_{m-1}(-j-1, 0, 0)_1$$
$$+ q^{2m}F_{m-1}(-j-1, -1, 0)_0]$$
$$= X_m^*(\rho_j, 3, 1; q^{-1}) + q^m[F_{m-1}(j, 0, 0)_1$$
$$+ q^m F_{m-1}(j, -1, 0)_0 - qF_{m-1}(j, -1, 1)_1$$
$$- q^{m-1}F_{m-1}(j, 0, 1)_0]$$
$$= X_m^*(\rho_j, 3, 1; q^{-1}) \qquad [\text{by } (3.10_b]$$

Hence $X_m^*(a, b, c; q^{-1})$ satisfies (3.5) and (3.6). Therefore (3.4) is valid.

We may now easily conclude with limits of $X_m(a, b, c; q^{-1})$, which are equivalent to (3.16a)–(3.16d) of II.

Recall

$$Q(q) = \prod_{j=1}^{\infty} (1 - q^j) \tag{3.13}$$

and

$$R(q) = \prod_{j=1}^{\infty} (1 - q^{2j-1}) = \prod_{j=1}^{\infty} \frac{1}{1 + q^j} \tag{3.14}$$

Then we may rewrite (2.53) and (2.54) as

$$\lim_{\substack{m \to \infty \\ m - A \equiv \lambda \ (\mathrm{mod}\ 2)}} T_0(m, A, q) = \frac{1}{2} [R(-q) + (-1)^\lambda R(q)] \tag{3.15}$$

Hence

$$\begin{aligned}
&\lim_{\substack{m - \infty \\ m - k + j \equiv \lambda \ (\mathrm{mod}\ 2)}} F_m(j, k, l)_0 \\
&\quad = \sum_{\substack{\mu = -\infty \\ \mu \text{ even}}}^{\infty} q^{35\mu^2 + (7k + 7l - 10j - 5)\mu + (j-k)(j-l)} \\
&\qquad \times \frac{1}{2Q(q^2)} [R(-q) + (-1)^\lambda R(q)] \\
&\qquad + \sum_{\substack{\mu = -\infty \\ \mu \text{ odd}}}^{\infty} q^{35\mu^2 + (7k + 7l - 10j - 5)\mu + (j-k)(j-l)} \\
&\qquad \times \frac{1}{2Q(q^2)} [R(-q) - (-1)^\lambda R(q)] \\
&\quad = \frac{R(-q)}{2Q(q^2)} \{35, 7k + 7l - 10j - 5, (j-k)(j-l); q^2\} \\
&\qquad + \frac{(-1)^\lambda R(-q)}{2Q(q^2)} \{35, 7k + 7l - 10j - 5, (j-k)(j-l); q^2\}_- \\
&\qquad [\text{in the notation of (2.13) and (2.14) in II}]
\end{aligned} \tag{3.16}$$

Also, by (2.51),

$$\lim_{m \to \infty} F_m(j, k, l)_1 = \frac{1}{Q(q^2)\, R(q^2)} \{35, 7k + 7l - 10j - 5, (j-k)(j-l); q^2\} \tag{3.17}$$

Applying these limits to (3.4), we find that

$$\lim_{m\to\infty} X_m(\rho_j, \rho_k, \rho_k; q^{-1})$$

$$= \frac{1}{Q(q^2)\,R(q^2)} [\{35, 14k-10j-5, (j-k)^2; q^2\} - q^{-2j-1}\{35, 14k+10j+5, (j+k+1)^2; q^2\}] \tag{3.18}$$

and for $l \neq k$

$$\lim_{\substack{m\to\infty \\ m-k+j\equiv\lambda (\text{mod } 2)}} X_m(\rho_j, \rho_k, \rho_l; q^{-1})$$

$$\begin{aligned} &= \frac{R(-q)}{2Q(q^2)} [\{35, 7k+7l-10j-5, (j-k)(j-l); q^2\} \\ &\quad - \{35, 7k+7l+10j+5, (j+k+1)(j+l+1); q^2\}] \\ &\quad + \frac{(-1)^\lambda R(q)}{2Q(q^2)} [\{35, 7k+7l-10j-5, (j-k)(j-l); q^2\}_- \\ &\quad - \{35, 7k+7l+10j+5, (j+k+1)(j+l+1); q^2\}_-] \end{aligned} \tag{3.19}$$

Inspection shows that Eqs. (3.18) and (3.19) provide the large-m results for $X_m(a, b, c; q^{-1})$ discussed in the section on regime III in II.

4. REGIME I

In this section we consider the polynomials of (2.6) in II:

$$X_m(a, b, c; q) = \sum_{\sigma_2} \cdots \sum_{\sigma_m} q^{\sum_{j=1}^{m} jH(\sigma_j, \sigma_{j+1}, \sigma_{j+2})} \tag{4.1}$$

under the conditions $0 \leqslant \sigma_j$, $0 \leqslant \sigma_j + \sigma_{j+1} \leqslant 2$, $1 \leqslant j \leqslant m+1$, $\sigma_1 = 3-a$, $\sigma_{m+1} = 3-b$, $\sigma_{m+2} = 3-c$, and

$$H(a, b, c) = b \tag{4.2}$$

i.e., condition (2.5b) of II. Subject to the above conditions, we see that $4-b \leqslant c \leqslant 3$, and otherwise c does not affect the polynomials. To emphasize the independence from c, we write

$$Y_m(a, b; q) = X_m(a, b, c; q) = \sum_{\sigma_2} \cdots \sum_{\sigma_m} q^{\sum j\sigma_{j+1}} \tag{4.3}$$

Inspection of (4.1) and the related summation conditions yields the following set of defining recurrences and initial conditions (with $1 \leqslant a, b \leqslant 3$):

$$Y_m(a, b; q) = q^{m(3-b)} \sum_{j=4-b}^{3} Y_{m-1}(a, j; q) \tag{4.4}$$

$$Y_0(a, b; q) = \begin{cases} 1 & \text{if} \quad a = b \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

Our object is to represent the $Y_m(a, b; q)$ in terms of q-trinomial coefficients. We define for ε, $\psi = 0$ or 1

$$\begin{aligned} \mathfrak{F}_m(j, k; \psi, \varepsilon; q) &= \mathfrak{F}_m(j, k; \psi, \varepsilon) \\ &= \sum_{\mu=-\infty}^{\infty} q^{14\mu^2 + (-4j+5-7\psi)\mu} \binom{m; 7\mu - j + k - \varepsilon; q}{7\mu - j + k}_2 \end{aligned} \tag{4.6}$$

The required representation is

$$\begin{aligned} Y_m(\rho_j, \rho_k; q) = {} & q^{m(1-\delta(1,k)) + \delta(0,k)j} \mathfrak{F}_m(j, k; \delta(0, k), \delta(2, k)) \\ & - q^{(2j+1)(1-\delta(0,k))} \mathfrak{F}_m(-j-1, k; \delta(0, k), \delta(2, k)) \end{aligned} \tag{4.7}$$

As in Section 3, we obtain recurrences for the $\mathfrak{F}_m(j, k; 4, \varepsilon)$ from q-trinomial recurrences.

From (2.28)

$$\begin{aligned} \mathfrak{F}_m(j, k; \psi, \varepsilon) = {} & \mathfrak{F}_{m-1}(j, k; \psi, \varepsilon) + q^{m-\varepsilon-1} \mathfrak{F}_{m-1}(j, k+1; \psi, \varepsilon+1) \\ & + q^{m+j-k} \mathfrak{F}_{m-1}(j, k-1; \psi+1, \varepsilon) \end{aligned} \tag{4.8}$$

From (2.29)

$$\begin{aligned} \mathfrak{F}_m(j, k; \psi, \varepsilon) = {} & \mathfrak{F}_{m-1}(j, k; \psi, \varepsilon) + q^{m+j-k} \mathfrak{F}_{m-1}(j, k-1; \psi+1, \varepsilon+1) \\ & + q^{m-j+k-\varepsilon} \mathfrak{F}_{m-1}(j, k+1; \psi-1, \varepsilon) \end{aligned} \tag{4.9}$$

From (2.25)

$$\begin{aligned} \mathfrak{F}_m(j, k; \psi, 1) = {} & q^{m-1} \mathfrak{F}_{m-1}(j, k; \psi, 1) + q^{k-j} \mathfrak{F}_{m-1}(j, k+1; \psi-1, 0) \\ & + \mathfrak{F}_{m-1}(j, k-1; \psi, 0) \end{aligned} \tag{4.10}$$

Replacing μ by $-\mu$ in (4.6) with $\varepsilon = 0$ or 1, we have

$$\mathfrak{F}_m(j, k; \psi, \varepsilon) = q^{\varepsilon(k-j)} \mathfrak{F}_m(-j-1, -k-1; 2+\varepsilon-\psi, \varepsilon) \tag{4.11}$$

Replacing μ by $-\mu-1$ in (4.6), we obtain

$$\mathfrak{F}_m(j,k;-1,0)=q^{4j+2}\mathfrak{F}_m(-j-1,k;-1,0) \tag{4.12}$$

Finally, from (2.27)

$$\begin{aligned}&\mathfrak{F}_m(j,k;\psi,0)+q^m\mathfrak{F}_m(j,k+1;\psi,1)\\&\quad-\mathfrak{F}_m(j,k+1;\psi,0)-q^{m+j-k}\mathfrak{F}_m(j,k;\psi+1,1)=0\end{aligned} \tag{4.13}$$

We let $y_m(\rho_j,\rho_k;q)$ denote the right-hand side of (4.7), and we now verify (4.4) and (4.5) for $y_m(\rho_j,\rho_k;q)$:

$$\begin{aligned}y_m(\rho_j,3;q)&=\mathfrak{F}_m(j,1;0,0)-q^{2j+1}\mathfrak{F}_m(-j-1,1;0,0)\\&=\mathfrak{F}_{m-1}(j,1;0,0)+q^{m-1}\mathfrak{F}_{m-1}(j,2;0,1)\\&\quad+q^{m+j-1}\mathfrak{F}_{m-1}(j,0;1,0)\\&\quad-q^{2j+1}[\mathfrak{F}_{m-1}(-j-1,1;0,0)+q^{m-1}\mathfrak{F}_{m-1}(-j-1,2;0,1)\\&\quad+q^{m-j-2}\mathfrak{F}_{m-1}(-j-1,0;1,0)]\qquad\text{[by (4.8)]}\\&=y_{m-1}(\rho_j,3;q)+y_{m-1}(\rho_j,2;q)+y_{m-1}(\rho_j,1;q)\end{aligned} \tag{4.14}$$

which is (4.4) for $b=3$.

$$\begin{aligned}y_m(\rho_j,2;q)&=q^m[\mathfrak{F}_m(j,2;0,1)-q^{2j+1}\mathfrak{F}_m(-j-1,2;0,1)]\\&=q^m\{q^{m-1}\mathfrak{F}_{m-1}(j,2;0,1)+q^{2-j}\mathfrak{F}_{m-1}(j,3;-1,0)\\&\quad+\mathfrak{F}_{m-1}(j,1;0,0)-q^{2j+1}[q^{m-1}\mathfrak{F}_{m-1}(-j-1,2;0,1)\\&\quad+q^{3+j}\mathfrak{F}_{m-1}(-j-1,3;-1,0)+\mathfrak{F}_{m-1}(-j-1,1;0,0)]\}\\&\quad\text{[by (4.10)]}\\&=q^m y_{m-1}(\rho_j,2;q)+q^{m+2-j}[\mathfrak{F}_{m-1}(j,3;-1,0)\\&\quad-q^{4j+2}\mathfrak{F}_{m-1}(-j-1,3;-1,0)]+q^m y_{m-1}(\rho_j,3;q)\\&=q^m[y_{m-1}(\rho_j,2;q)+y_{m-1}(\rho_j,3;q)]\qquad\text{[by (4.12)]}\end{aligned} \tag{4.15}$$

which is (4.4) for $b=2$.

$$\begin{aligned}y_m(\rho_j,1;q)&=q^{m+j}[\mathfrak{F}_m(j,0;1,0)-\mathfrak{F}_m(-j-1,0;1,0)]\\&=q^{m+j}\{\mathfrak{F}_{m-1}(j,0;1,0)+q^{m+j}\mathfrak{F}_{m-1}(j,-1;2,1)\\&\quad+q^{m-j}\mathfrak{F}_{m-1}(j,1;0,0)-[\mathfrak{F}_{m-1}(-j-1,0;1,0)\\&\quad+q^{m-j-1}\mathfrak{F}_{m-1}(-j-1,-1;2,1)\\&\quad+q^{m+j+1}\mathfrak{F}_{m-1}(-j-1,1;0,0)]\}\end{aligned}$$

$$\begin{aligned}
&= q^{2m} y_{m-1}(\rho_j, 3; q) + q^{m+j}[\mathfrak{F}_{m-1}(-j-1, -1; 1, 0) \\
&\quad + q^{m-1}\mathfrak{F}_{m-1}(-j-1, 0; 1, 1) - \mathfrak{F}_{m-1}(-j-1, 0; 1, 0) \\
&\quad - q^{m-j-1}\mathfrak{F}_{m-1}(-j-1, -1; 2, 1)] \\
&= q^{2m} y_{m-1}(\rho_j, 3; q) \qquad [\text{by (4.13)}]
\end{aligned} \tag{4.16}$$

which is (4.4) for $b=1$. Thus, (4.4) is established for the $y_m(\rho_j, \rho_k; q)$.

As for (4.5) for $y_0(\rho_j, \rho_k; q)$, we see that this follows immediately from the fact that $\mathfrak{F}_0(j, k; \psi, \varepsilon) = \delta(j, k)$.

Thus, (4.7) is established.

It is an easy consequence of (4.4) that $q^{m(b-3)}Y_m(a, b; q)$ does not depend on b in the limit. Hence,

$$\begin{aligned}
&\lim_{m\to\infty} q^{m(b-3)} Y_m(\rho_j, b; q) \\
&\quad = \lim_{m\to\infty} Y_m(\rho_j, 3; q) \\
&\quad = \frac{1}{Q(q)}\left(\sum_{\mu=-\infty}^{\infty} q^{14\mu^2-(4j-5)\mu} - q^{2j+1}\sum_{\mu=-\infty}^{\infty} q^{14\mu^2+(4j+9)\mu}\right) \\
&\qquad [\text{by (4.7) and (2.48)}] \\
&\quad = \prod_{\substack{n=1 \\ n\not\equiv 0, \pm\rho_j \,(\mathrm{mod}\ 7)}}^{\infty} (1-q^n)^{-1}
\end{aligned} \tag{4.17}$$

where the last line follows by Jacobi's triple product identity.

5. REGIME II

As was pointed out in II, we regain for regime II in our new model the same limits as in our 8VSOS model.[6] We prove this is the case with a series of propositions.

Proposition 5.1. Let $d_m(j, k, l)$ denote the degree of the polynomial $X_m(\rho_j, \rho_k, \rho_l; q^{-1})$. Then, for $-2 \leqslant \alpha \leqslant 2$,

$$d_{5m+\alpha}(j, k, l) = 15m^2 + [6\alpha + kl + (l-1)(4l-3k-3)]\, m + d_\alpha(j, k, l) \tag{5.1}$$

Proof. This is merely a trivial but tedious mathematical induction using the recurrence (3.6). We omit the details. ∎

Proposition 5.2. For $j=0$, 1, or 2

$$q^{m^2-j}X_m(\rho_j, 1, 3; q) = \sum_{\mu=-\infty}^{\infty} q^{14\mu^2-(4j+2)\mu}[t_0(m, 7\mu-\alpha, q)-t_0(m, 7\mu-\alpha-1, q)] \tag{5.2}$$

Proof. By (3.4) and (3.1)

$$X_m(\rho_j, 1, 3; q^{-1}) = F_m(j, 0, 1) - q^{-2j-1}F_m(-j-1, 0, 1)$$
$$= \sum_{\mu=-\infty}^{\infty} q^{35\mu^2+(2-10j)\mu+j(j-1)}T_0(m, 7\mu-j, q)$$
$$- \sum_{\mu=-\infty}^{\infty} q^{35\mu^2-(12+10j)\mu+j^2+j+1}T_0(m, 7\mu-j-1, q)$$

where we have replaced μ by $-\mu$ in the second sum. Consequently, replacing q by q^{-1}, we obtain

$$q^{m^2-j}X_m(\rho_j, 1, 3; q)$$
$$= q^{m^2-j}\left\{ \sum_{\mu=-\infty}^{\infty} q^{-35\mu^2-(2-10j)\mu-j^2+j+(7\mu-j)^2-m^2}t_0(m, 7\mu-j, q) \right.$$
$$\left. - \sum_{\mu=-\infty}^{\infty} q^{-35\mu^2+(10j+12)\mu-j^2-j-1+(7\mu-j-1)^2-m^2}t_0(m, 7\mu-j, q) \right\}$$

[by (2.33_0)]

$$= \sum_{\mu=-\infty}^{\infty} q^{14\mu^2-(4j+2)\mu}[t_0(m, 7\mu-j, q)-t_0(m, 7\mu-j-1, q)] \tag{5.3}$$

as desired. ∎

Now we must recall some polynomial definitions (Ref. 6, §2.6):

$$x_m(a, b, c) = q^{(m+a-b)/4}[f_m(a, b, c) - f_m(-a, b, c)] \tag{5.4}$$

$$f_m(a, b, c) = \sum_{\lambda=-\infty}^{\infty} q^{7\lambda^2-a\lambda+(b+1-c)(14\lambda+b-a)/4} \begin{bmatrix} m \\ \frac{1}{2}(m+a-b)-7\lambda \end{bmatrix} \tag{5.5}$$

$$x_m(a, 1, 2) = q^{(m-1)(m+6)/20}\hat{x}_m(a, 1, 2) \tag{5.6}$$

Proposition 5.3. For $\alpha=0, \pm 1, \pm 2$,

$$\lim_{\substack{m\to\infty \\ m\equiv\alpha \,(\mathrm{mod}\, 5)}} q^{\binom{m+1}{2}-\binom{j}{2}-(2m-1)(m+3)/10}x_m(\rho_j, 1, 3; q^{1/2})$$
$$= \lim_{\substack{m\to\infty \\ m\equiv\alpha \,(\mathrm{mod}\, 5)}} \hat{x}_{2m}(2j+1, 1, 3) \tag{5.7}$$

Proof. By Proposition 5.2 and (2.34_0) with $j=0$, 1, or 2

$$q^{(m^2-j)/2}X_m(\rho_j, 1, 3; q^{1/2})$$

$$= \sum_{\mu=-\infty}^{\infty} q^{7\mu^2-(2j+1)\mu} \left\{ \binom{m; 7\mu-\alpha; q}{7\mu-\alpha}_2 - \binom{m; 7\mu-\alpha-1; q}{7\mu-\alpha-1}_2 \right\}$$

$$= \sum_{h=0}^{m} (-1)^h q^{mh-\binom{h+1}{2}} \begin{bmatrix} m \\ h \end{bmatrix}$$

$$\times \left\{ \sum_{\mu=-\infty}^{\infty} q^{7\mu^2-(2j+1)\mu} \left(\begin{bmatrix} 2m-2h \\ m-h+7\mu-j \end{bmatrix} - \begin{bmatrix} 2m-2h \\ m-h+7\mu-j-1 \end{bmatrix} \right) \right.$$

[by (2.35)]

$$= \sum_{h=0}^{m} (-1)^h q^{mh-\binom{h+1}{2}} \begin{bmatrix} m \\ h \end{bmatrix} q^{-(m-h+j)/2} x_{2m-2h}(2j+1, 1, 2)$$

[by (5.4)]

$$= \sum_{h=0}^{m} (-1)^h q^{mh-\binom{h+1}{2}} \begin{bmatrix} m \\ h \end{bmatrix} q^{-(m-h+j)/2+(2m-2h-1)(m-h+3)/10}$$

$$\times \hat{x}_{2m-2h}(2j+1, 1, 2) \qquad \text{[by (5.6)]} \tag{5.8}$$

Consequently,

$$q^{(m^2-j)/2+(m+j)/2-(2m-1)(m+3)/10} X_m(\rho_j, 1, 3; q^{1/2})$$

$$= \sum_{h=0}^{m} (-1)^h q^{3h(2m-h)/10-7h/10} \begin{bmatrix} m \\ h \end{bmatrix} \hat{x}_{2m-2h}(2j+1, 1, 2) \tag{5.9}$$

Now in (5.9) we note two things: (1) $\hat{x}_{2m-2h}(2j+1, 1, 2)$ converges to a nonzero limit as m passes to infinity in an arithmetic progression of difference 5 (Ref. 6, §2.6), and (2) the exponent on q in the sum is at least $(3m-5)/5$ for $1 \leqslant h \leqslant m$. Therefore,

$$\lim_{\substack{m\to\infty \\ m\equiv\alpha \pmod 5}} q^{\binom{m+1}{2}-(2m-1)(m+3)/10} X_m(\rho_j, 1, 3; q^{1/2})$$

$$= \lim_{\substack{m\to\infty \\ m\equiv\alpha \pmod 5}} \hat{x}_{2m}(2j+1, 1, 2) \quad \blacksquare \tag{5.10}$$

Finally, we note that in order to know the limiting values of the various $X_m(\rho_j, \rho_k, \rho_l; q^{1/2})$, we need only know those of $X_m(\rho_j, 1, 3; q^{1/2})$. This is because by (3.6) with $b=3$ and $c=1$ we have $X_m(\rho_j, 3, 1; q^{1/2})$; then by (3.6) with $b=2$ and $c=3$ we have $X_m(\rho_j, 3, 2; q^{1/2})$; then, by (3.6) with

$b=c=3$ we have $X_m(\rho_j, 3, 3; q^{1/2})$; then, by (3.6) with $b=3$ and $c=2$ we have $X_m(\rho_j, 3, 2; q^{1/2})$, and finally by (3.6) with $b=c=2$ we have $X_m(\rho_j, 2, 2; q^{1/2})$. These observations are made merely by examining each instance of (3.6) and using Proposition 5.1 to determine which term actually contributes the highest power of q.

Consequently, Proposition 5.3 establishes results equivalent to (3.4)–(3.7) of II.

6. REGIME IV

Now we must consider polynomials reciprocal to those of regime I, i.e., we need to consider $\mathfrak{F}_m(j, k, \psi, \varepsilon; q^{-1})$ from (4.6). By (2.60) and (2.61) with $\lambda=0$ or 1

$$\lim_{\substack{m \to \infty \\ m \equiv \lambda \ (\mathrm{mod}\ 2)}} q^{(m^2-\lambda)/2}\mathfrak{F}_m(j, k; \psi, 0; q^{-1})$$

$$= \frac{R(-q^{1/2})}{2Q(q)}\{21, -6j+10-14\psi+14k, (j-k)^2; q\}$$

$$+\frac{R(q^{1/2})(-1)^{j-k-\lambda}}{2Q(q)}\{21, -6j+10-14\psi+14k, (j-k)^2; q\}_- \tag{6.1}$$

By (2.51), (2.33), and (2.34),

$$\lim_{m \to \infty} q^{\binom{m}{2}}\mathfrak{F}_m(j, k; \psi, 1; q^{-1})$$

$$= \frac{q^{\binom{k-j}{2}}}{Q(q)\,R(q)}\{21, -6j+3-14\psi+14k, (j-k)^2+(j-k); q\} \tag{6.2}$$

We merely substitute these limits into the limiting case of (4.7) in order to obtain the limits for regime IV.

First we note the reciprocal polynomials for those in regime I are defined by

$$y_{2m}(a, b; q) \equiv q^{2m^2+(3-b)m+\min(0,a-b)}Y_{2m}(a, b; q^{-1}) \tag{6.3}$$

$$y_{2m+1}(a, b; q) \equiv q^{2m^2-(b-1)m+\min(0,a+b-4)}Y_{2m-1}(a, b; q^{-1}) \tag{6.4}$$

The limits are

$$\lim_{m \to \infty} y_m(\rho_j, 2; q)$$

$$= \frac{1}{Q(q)\,R(q)}(\{21, 11-6j, 0; q\} - \{21, 17+6j, 2j^2+2; q\}) \tag{6.5}$$

$$\lim_{m\to\infty} y_{2m}(\rho_j, 3; q)$$

$$= \lim_{m\to\infty} y_{2m+1}(\rho_j, 1; q)$$

$$= \frac{R(-q^{1/2})}{2Q(q)} [\{21, -4+6j, (j-1)(3-2j); q\}$$

$$-\{21, 32-6j, -2j^2+j+9; q\}]$$

$$-\frac{(-1)^j R(q^{1/2})}{2Q(q)} [\{21, -4+6j, (j-1)(3-2j); q\}_-$$

$$-\{21, 32-6j, -2j^2+j+9; q\}_-] \tag{6.6}$$

and

$$\lim_{m\to\infty} y_{2m}(\rho_j, 1; q)$$

$$= \lim_{m\to\infty} y_{2m+1}(\rho_j, 3; q)$$

$$= \frac{R(-q^{1/2})}{2Q(q)} [\{21, -4+6j, j(j-2); q\}$$

$$-\{21, 10+6j, j^2+1; q\}]$$

$$+\frac{(-1)^j R(q^{1/2})}{2Q(q)} [\{21, -4+6j, j(j-2); q\}_-$$

$$+\{21, 10+6j, j^2+1; q\}_-] \tag{6.7}$$

These results are equivalent to (3.20a) and (3.20b).

7. AN ALTERNATIVE APPROACH TO REGIMES I AND IV

As we remarked in II, we have an alternative treatment of regimes I and IV, which yields different expressions for the limits in (6.5)–(6.7). It appears at this time that the uniformity provided by our q-trinomial coefficients will extend to the models with more than two particles per site. However, our alternative method may turn out to have other applications, so we describe it briefly.

We start with a two-variable generating function for polynomials $D_{r,s}(a;q)$:

$$\sum_{r,s\geqslant 0} D_{r,s}(a;q)\,x^r y^s$$
$$\equiv f_a(x,y)$$
$$= \sum_{i,j\geqslant 0} \frac{q^{(i+j)^2+i^2+(3-a)i+\delta(1,a)j}x^{4i+2j}y^{2i+2j}}{(x;q)_{i+1}\,(y;q)_{j+1}} \tag{7.1}$$

It is then easy to find simple functional equations for $f_a(x,y)$ that directly imply

$$D_{r,s}(a;q)-D_{r-1,s}(a;q)=q^{r+1-a}D_{r-4,s-2}(a;q) \tag{7.2}$$

provided r is odd or $r>s$; and

$$D_{r,s}(a;q)-D_{r,s-1}(a;q)=q^{s-1+\delta(1,a)}D_{r-2,s-2}(a;q) \tag{7.3}$$

provided s is odd or $s>r/2$.

Using these recurrences as well as (4.4) and (4.5), we find for $m\geqslant 1$

$$Y_m(a,b;q)=q^{(3-b)m}D_{2m+6-b,m-\delta(1,a)-\delta(1,b)}(a;q) \tag{7.4}$$

The $D_{r,s}(a;q)$ can also in certain instances be identified with certain polynomials whose limiting behavior is easily discerned:

$$\Delta_{K,i}(a,b;q)=\sum_{\mu=-\infty} q^{\mu(2K\mu-K+2i)}\begin{bmatrix} a+b \\ a-K\mu \end{bmatrix}$$
$$-\sum_{\mu=-\infty}^{\infty} q^{(2\mu-1)(K\mu-i)}\begin{bmatrix} a+b \\ a-K\mu+i \end{bmatrix} \tag{7.5}$$

These polynomials appear in Ref. 18 [p. 56, Eq. (2.10) corrected] as $D_{K,i}(0;a,b;q)$ and in Ref. 19 as $\delta_{K,i}(a,b;1,1)$. Again recurrences of the $\Delta_{K,i}(a,b;q)$ plus (7.2) and (7.3) allow us to prove for $-1\leqslant a-b\leqslant 2$

$$D_{a+b-[|a-b-1/2|],a+b}(3,q)=\Delta_{7,3}(a,b;q) \tag{7.6}$$

$$D_{2b-\lambda(3\lambda-1)/2,a+b}(\alpha;q)+\delta(-1,a-b)\,q^{b^2+(2-\alpha)m}=\Delta_{7,\alpha}(a-\alpha+3,b;q) \tag{7.7}$$

where $\alpha=1$ or 2 and $\lambda=a-b$.

It is then possible to relate the $\Delta_{7,\alpha}(a, b; q)$ to the $Y_m(a, b; q)$ using the following easily established identities for the $D_{r,s}(a; q)$:

$$D_{M+R,M}(a; q) = \sum_{0 \leqslant 2j \leqslant R} q^{j(M+3-a)} \begin{bmatrix} R-j \\ j \end{bmatrix} D_{M-2j,M-2j}(a; q) + \sum_{1 \leqslant 2j \leqslant R+1} q^{j(M+2-a)} \begin{bmatrix} R-j \\ j-1 \end{bmatrix} D_{M-2j-1,M-2j}(a; q) \quad (7.8)$$

and for $a = 1$ or 2

$$D_{M+R,M}(a; q) = \sum_{0 \leqslant 2j \leqslant R} q^{(j+1)(M+1-a)} \begin{bmatrix} R-j \\ j \end{bmatrix} \Big\{ D_{m-2j-4,M-2j-2}(a; q) + \frac{1}{2} [1 + (-1)^M]\, q^{\left(\frac{M-2j-2}{2}\right)^2 + \delta(1,a)\left(\frac{M-2j-2}{2}\right)} \Big\} + \sum_{0 \leqslant 2j \leqslant R+1} q^{j(M+2-a)} \begin{bmatrix} R+1-j \\ j \end{bmatrix} D_{M-2j-1,M-2j}(a; q) \quad (7.9)$$

This concludes the full description of our alternative approach. We next provide a prototypical example of how the above actually provides useful representations of the $Y_m(a, b; q)$:

$$Y_{2m}(3, 3; q) = \sum_{j=0}^{m} q^{2mj} \begin{bmatrix} 2m-j \\ j \end{bmatrix} \Delta_{7,3}(m-j, m-j; q) + \sum_{j=1}^{m-1} q^{(2m-1)j} \begin{bmatrix} 2m-j \\ j-1 \end{bmatrix} \Delta_{7,3}(m-j+1, m-j-1; q) \quad (7.10)$$

The regime IV result then proceeds as follows:

$$\begin{aligned} &\lim_{m \to \infty} y_{2m}(3, 3; q) \\ &\quad = \lim_{m \to \infty} q^{2m^2} Y_{2m}(3, 3; q^{-1}) \\ &\quad = \lim_{m \to \infty} q^{2m^2} \Big\{ \sum_{j=0}^{m} q^{-2j(m-j)} \begin{bmatrix} m+j \\ 2j \end{bmatrix} q^{-2j(m-j)} \Delta_{7,3}(j, j; q^{-1}) \\ &\quad\quad + \sum_{j=0}^{m-1} q^{-(2m-1)(m-j)} \begin{bmatrix} m+j \\ 2j+1 \end{bmatrix} q^{-(2j+1)(m-j-1)} \Delta_{7,3}(j+1, j-1; q^{-1}) \end{aligned} \quad (7.11)$$

We now replace $\Delta_{7,3}(j+1, j-1; q^{-1})$ by $\Delta_{7,3}(j+2, j-1; q^{-1})$, since these polynomials turn out to be identical. Hence

$$\begin{aligned}
\lim_{m\to\infty} & y_{2m}(3, 3; q) \\
&= \sum_{j=0}^{\infty} q^{2j^2} \left\{ q^{-j^2} \sum_{\mu=-\infty}^{\infty} \frac{q^{35\mu^2+\mu}}{(q)_{j-7\mu}(q)_{j+7\mu}} \right. \\
&\quad \left. - q^{-j^2+6} \sum_{\mu=-\infty}^{\infty} \frac{q^{35\mu^2-29\mu}}{(q)_{j-7\mu+3}(q)_{j+7\mu-3}} \right\} \\
&\quad + \sum_{j=0}^{\infty} q^{2j^2+2j+1} \left\{ q^{-(j+2)(j-1)} \sum_{\mu=-\infty}^{\infty} \frac{q^{35\mu^2-20\mu}}{(q)_{j+2-7\mu}(q)_{j+7\mu-1}} \right. \\
&\quad \left. - q^{-j^2-j+17} \sum_{\mu=-\infty}^{\infty} \frac{q^{35\mu^2-50\mu}}{(q)_{j+5-7\mu}(q)_{j+7\mu-4}} \right\} \\
&= \sum_{\mu=-\infty}^{\infty} q^{35\mu^2+\mu} \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_{j-7\mu}(q)_{j+7\mu}} \\
&\quad - \sum_{\mu=-\infty}^{\infty} q^{35\mu^2-29\mu+6} \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_{j-7\mu+3}(q)_{j+7\mu-3}} \\
&\quad + \sum_{\mu=-\infty}^{\infty} q^{35\mu^2-20\mu+3} \sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q)_{j+2-7\mu}(q)_{j+7\mu-1}} \\
&\quad - \sum_{\mu=-\infty}^{\infty} q^{35\mu^2-50\mu+18} \sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q)_{j+5-7\mu}(q)_{j+7\mu-4}}
\end{aligned} \tag{7.12}$$

The series with index j are all summable by (2.57):

$$\sum_{j=0}^{\infty} \frac{q^{j^2+(B-A)j}}{(q)_{j-A}(q)_{j+B}} = \frac{q^{AB}}{(q)_\infty} = \frac{q^{AB}}{Q(q)} \tag{7.13}$$

Hence

$$\begin{aligned}
\lim_{m\to\infty} & y_{2m}(3, 3; q) \\
&= \frac{1}{Q(q)} \left\{ \sum_{\mu=-\infty}^{\infty} q^{84\mu^2+\mu} + \sum_{\mu=-\infty}^{\infty} q^{84\mu^2-41\mu+5} \right. \\
&\quad \left. - \sum_{\mu=-\infty}^{\infty} q^{84\mu^2-71\mu+15} - \sum_{\mu=-\infty}^{\infty} q^{84\mu^2-113\mu+38} \right\}
\end{aligned} \tag{7.14}$$

As is obvious, (7.14) is quite different from the $j=1$ case of (6.6). Of course, all the regime IV limits may be done in this manner, and we find

$$\lim_{m\to\infty} y_m(j,2;q) = \frac{1}{Q(q)}[\{84, 28-12j, 0; q^2\} + \{84, 56-12j, 7-2j; q^2\} - \{84, 28+12j, 4j; q^2\} - \{84, 56+12j, 7+6j; q^2\}] \tag{7.15}$$

$$\lim_{m\to\infty} y_{2m+\delta(2,j)}(j,3;q) = \lim_{m\to\infty} y_{2m+\delta(2,j)-1}(j,1;q) = \frac{1}{Q(q)}[\{84, 35-12j, 0; q^2\} + \{84, 77-12j, 14-3j; q^2\} - \{84, 35+12j, 5j; q^2\} - \{84, 91-12j, 21-4j; q^2\}] \tag{7.16}$$

$$\lim_{m\to\infty} y_{2m+\delta(2,j)}(j,1;q) = \lim_{m\to\infty} y_{2m+\delta(2,j)-1}(j,3;q) = \frac{1}{Q(q)}[\{84, 12j-7, (j-1)(j-2); q^2\} + \{84, 49-12j, (3-j)^2; q^2\} - \{84, 12j+7, (j-1)^2+1; q^2\} - \{84, 49+12j, -4j^2+26j-21; q\}] \tag{7.17}$$

As we remarked in II, we have a direct proof of the equivalence of (6.5)–(6.7) with (7.15)–(7.17). The proof is somewhat intricate; the primary result at work is (see Ref. 20, p. 921 for notation)

$$\theta_2\left(u-\frac{\pi}{3}, q\right) - \theta_2\left(u+\frac{\pi}{3}, q\right) = \frac{\sqrt{3}\,\theta_1(u, q^3)\,\theta_3(u, q^3)\,\theta_4(u, q^3)}{q^{1/4}Q^2(q^3)} \tag{7.18}$$

We have not seen this result before; it may be verified by applying Liouville's theorem to the left-hand side divided by the right-hand side.

8. CONCLUSION

The next objective is to extend our work from the $n=3$ case to arbitrary n. The mathematical tools for this project clearly appear to be q-analogs of n-polynomial coefficients, i.e., q-analogs of the coefficients in

$$(1+x+x^2+\cdots+x^{n-1})^N \tag{8.1}$$

We also mention that the mapping ρ_j defined by (3.3) should extend as follows:

$$\rho_j = \begin{cases} 2j+1, & 0 \leqslant 2j < n \\ 2n-2j, & n \leqslant 2j < 2n \end{cases} \tag{8.2}$$

and so

$$\rho^{-1}(j) = \begin{cases} (j-1)/2, & j \text{ odd} \\ n-j/2, & j \text{ even} \end{cases} \tag{8.3}$$

This notation allows us to write (2.5a) of II as

$$H(n-a, n-b, n-c) = -|\rho^{-1}(a) - \rho^{-1}(c)| \tag{8.4}$$

and this appears to be the appropriate way to generalize succinctly regime III.

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